

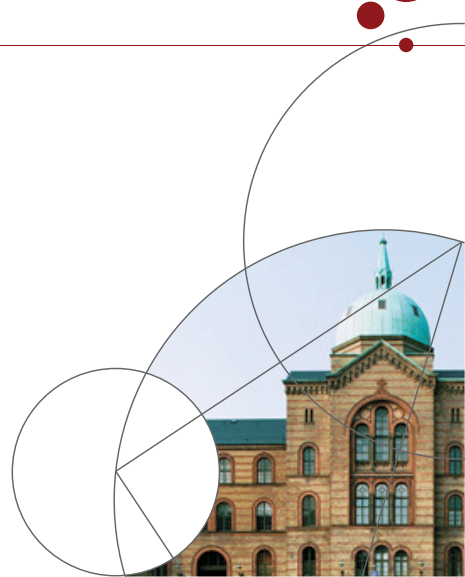


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Mikro II, lecture 8b

A short introduction to game theory

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Plan for the lecture

- 1 Definition of a two-player game
- 2 Best responses, Nash equilibria
- 3 Examples of games and equilibria
- 4 Games with many players



Equilibrium so far

- Most of our models include agents who make decisions individually but interact with each other (trades, smokers)
- Our agents' interaction has so far (almost) only been through *perfect competition / price-taking* ...
 - ① Agents' decisions depend only on their own situation and the market prices are taken for granted
 - ② Equilibrium is a situation in which prices have adjusted such that all decisions of optimizing agents comply with constraints. Interaction between the agents occurs *exclusively* via prices.
- ... or via *monopoly / Principal Agent*:
 - ① Monopolist / Principal has (market) power: sets price / offers contract
 - ② The interaction happens *solely* by allowing the Monopolist / Principal to anticipate other agents' reaction to the price / contract



Game Theory I

- We now want to analyze situations where price-taking does not make sense and where there is not one agent who has “all the power”
 - A small number of companies compete in a market: *Oligopoly* (oligo = few, polein = to sell), e.g. aircraft production (Boeing vs. Airbus)
 - Externalities revisited: There exist no market and price for a good (smoke, alarm, sheep)
 - **Games**: Rock-paper-scissors, chess, ludo, etc.
- Game theory is about strategic interactions: my actions affect you directly and vice versa, *and we both take this into account in our decisions.*



Socratic Quiz Question

Which of the following does usually not involve strategic interactions?

- a) Wage bargaining between a union and a firm.
- b) Trading cars in the used car market
- c) Monopsony
- d) Public goods
- e) Don't know



Game Theory II

- Game theory is covered in detail in Microeconomics III, so we only go through a short introduction.
- In particular, we will only focus on *simultaneous* games with *perfect information*
 - Perfect information: There is no inherent randomness (uncertainty) involved in the game: i.e. NOT ludo (dice)
 - Simultaneous Games: Players make their choices simultaneously (and only once): i.e rock paper scissors without repetition, but NOT chess.
- In more technical terms: We will only focus on so-called *pure strategies* (we will not allow players to act randomly)



A two-player game

Two-player game, definition

A game with two players, 1 and 2, consists of:

- A strategy set of possible strategies for each player, S_1 and S_2
- Utility functions that indicate each player's utility as a function of the strategies selected, $u_1(s_1, s_2)$ and $u_2(s_2, s_1)$, thus:

$$u_1 : S_1 \times S_2 \rightarrow \mathcal{R}$$

$$u_2 : S_2 \times S_1 \rightarrow \mathcal{R}$$



Strategy set

- The strategy sets S_1, S_2 indicate what players can choose to do; may be discrete or continuous
- Example 1: Rock paper scissors

$$S_1 = S_2 = \{rock, paper, scissors\}$$

- Example 2: Jeppe Druedahl (player 1) and Johannes (player 2) will meet at Nørrebrogade but have not agreed how far down we are going to meet up; where is each of us going?

$$S_1 = S_2 = [0, 1000]$$

(Assumption: Nørrebrogade is 1000 m long)



Utility Functions

- The utility functions u_1 and u_2 indicate how the utility of each players depends on what BOTH players do:

- Example 1: Rock paper scissor, the benefit of winning is 1

$$u_1(s_1, s_2) = \begin{cases} 1 & \text{for } (s_1, s_2) \in \{(rock, scissor), (scissor, paper), (paper, rock)\} \\ 0 & \text{otherwise} \end{cases}$$

$$u_2(s_2, s_1) = \begin{cases} 1 & \text{for } (s_2, s_1) \in \{(rock, scissor), (scissor, paper), (paper, rock)\} \\ 0 & \text{otherwise} \end{cases}$$

- Example 2: Jeppe's and my utility are greater the closer we end up:

$$u_1(s_1, s_2) = u_2(s_1, s_2) = -|s_1 - s_2|$$



New example: The driving game

- Two motorists drive on the same road and are approaching each other from opposite directions. They must choose whether to drive on the right or on the left side of the road
- They drive into each other if they do not both drive on the left or both drive on the right, giving them utility -1 . If they do not drive into each other they get utility 0 :

$$S_1 = S_2 = \{left, right\}$$

$$u_1(s_1, s_2) = u_2(s_2, s_1) = \begin{cases} 0 & \text{for } (s_1 = s_2) \\ -1 & \text{for } (s_1 \neq s_2) \end{cases}$$

- 2-player games with discrete strategy amounts can conveniently be written up as matrices.



Car game in matrix form

	2 plays <i>left</i>	2 plays <i>right</i>
1 plays <i>left</i>	0,0	-1,-1
1 plays <i>right</i>	-1,-1	0,0



Rock paper scissors in matrix form

	2 plays <i>rock</i>	2 plays <i>scissor</i>	2 plays <i>paper</i>
1 plays <i>rock</i>	0, 0	1, 0	0, 1
1 plays <i>scissor</i>	0, 1	0, 0	1, 0
1 plays <i>paper</i>	1, 0	0, 1	0, 0



Best response function

Best response function, player 1

A best response function for player 1, $s_1^*(s_2)$, indicates an optimal strategy for player 1 as a function of player 2's strategy, i.e.

$$s_1^* : S_2 \rightarrow S_1$$

and $s_1^*(s_2)$ solves the problem

$$\max_{s_1} u_1(s_1, s_2)$$

- First step towards equilibrium: Player 1 takes player 2's choice s_2 as given and maximizes utility $\Rightarrow$ solution is the best response (function of s_2)
- Best response for player 2 is defined as $s_2^*(s_1)$.



Car game, best response I

- Best response is easy in the car game: always optimally running on the same side as the other driver:

$$s_1^*(s_2) = s_2 \qquad s_2^*(s_1) = s_1$$

- Can be found conveniently looking at the matrix (sometimes called the “underline method”):



Car game, best response II

	2 plays <i>left</i>	2 plays <i>right</i>
1 plays <i>left</i>	<u>0</u> , <u>0</u>	-1, -1
1 plays <i>right</i>	-1, -1	<u>0</u> , <u>0</u>



Socratic Quiz question

What are the best responses in the following game?

	2 plays <i>opera</i>	2 plays <i>football</i>
1 plays <i>opera</i>	3, 2	1, 1
1 plays <i>football</i>	0, 0	2, 3

- a) Player 1: *opera* if 2 plays *opera*, *opera* if 2 plays *football*. Player 2: *opera* if 1 plays *opera*, *football* if 1 plays *football*.
- b) Player 1: *football* if 2 plays *opera*, *football* if 2 plays *football*. Player 2: *football* if 1 plays *opera*, *football* if 1 plays *football*.
- c) Player 1: *opera* if 2 plays *opera*, *football* if 2 plays *football*. Player 2: *opera* if 1 plays *opera*, *football* if 1 plays *football*.
- d) Player 1: *football* if 2 plays *opera*, *opera* if 2 plays *football*. Player 2: *opera* if 1 plays *opera*, *football* if 1 plays *football*.
- e) Don't know.



Nash equilibrium

Nash equilibrium

A Nash equilibrium is a set of strategies for the two players $(\bar{s}_1, \bar{s}_2) \in S_1 \times S_2$ which are best responses to each other, i.e.:

$$s_1^*(\bar{s}_2) = \bar{s}_1$$

$$s_2^*(\bar{s}_1) = \bar{s}_2$$

where s_1^* is a best response function for player 1 and s_2^* is a best response function for player 2. It follows that in a Nash equilibrium each player has maximized utility given what the other player does:

$$u(\bar{s}_1, \bar{s}_2) = \max_{s_1} u(s_1, \bar{s}_2)$$

$$u(\bar{s}_2, \bar{s}_1) = \max_{s_2} u(s_2, \bar{s}_1)$$



Nash equilibrium, discussion



- This equilibrium concept, named after John Nash (1994 Nobel Prize), forms the fundamental basis of game theory.
- The Nash equilibrium is obviously a stable situation: In a Nash equilibrium, I cannot make myself better off by changing my behavior (same for opponent).
- How do we get into equilibrium? Similarly as for the Walrasian price equilibrium, it is not entirely clear. The hope / assumption is that in practice we will often end up near the stable situation.



New example I: Prisoner's Dilemma

- Let's try to set up a new example and solve for a Nash equilibrium from scratch; consider the game defined by the following matrix:

	2 plays C	2 plays D
1 plays C	3, 4	-3, 6
1 plays D	5, -2	0, 0

- We will start by finding the best responses as before.



New example II: Prisoner's Dilemma

	2 plays C	2 plays D
1 plays C	3, 4	-3, <u>6</u>
1 plays D	<u>5</u> , -2	<u>0</u> , <u>0</u>

- Nash equilibrium is defined as two strategies that are best responses to each other $\Rightarrow$ a cell where both numbers are underlined.
- The Nash equilibrium here is that both players play D (formally: $\bar{s}_1, \bar{s}_2 = (D, D)$).



New example III: Prisoner's Dilemma

	2 plays C	2 plays D
1 plays C	3, 4	-3, <u>6</u>
1 plays D	<u>5</u> , -2	<u>0</u> , <u>0</u>

- Note that the example here already illustrates that Nash equilibrium is *not necessarily* efficient: C, C gives both players significantly higher utility than the NE (called *Prisoner's Dilemma*)
- But C, C can never be an equilibrium: When player 1 makes her individual decision, she would try to jump down to D, C (best response) (player 2 would jump to C, D)



Car game, Nash equilibrium

	2 plays <i>left</i>	2 plays <i>right</i>
1 plays <i>left</i>	<u>0</u> , <u>0</u>	-1, -1
1 plays <i>right</i>	-1, -1	<u>0</u> , <u>0</u>

- New Insight about Nash Equilibrium: There can be more than one! Here, both (*left, left*) and (*right, right*) are equilibria (the game is called a *coordination game*)
- Empirically, different countries have coordinated on different equilibria (e.g. UK vs Denmark).



Socratic Quiz question

How many Nash equilibria are there in the following game?

	2 plays <i>opera</i>	2 plays <i>football</i>
1 plays <i>opera</i>	3, 2	1, 1
1 plays <i>football</i>	0, 0	2, 3

- a) None.
- b) One.
- c) Two.
- d) Three.
- e) Don't know.



Rock-paper-scissor, Nash Equilibrium

- Let's try to find Nash equilibria in rock-paper-scissor
- We can find the best responses just as before; only difference is that there are three possible strategies for each player

	2 plays <i>rock</i>	2 plays <i>scissor</i>	2 plays <i>paper</i>
1 plays <i>rock</i>	0, 0	1, 0	0, 1
1 plays <i>scissor</i>	0, 1	0, 0	1, 0
1 plays <i>paper</i>	1, 0	0, 1	0, 0



Rock-paper-scissor, Nash equilibrium

	2 plays <i>rock</i>	2 plays <i>scissor</i>	2 plays <i>paper</i>
1 plays <i>rock</i>	0, 0	<u>1</u> , 0	0, <u>1</u>
1 plays <i>scissor</i>	0, <u>1</u>	0, 0	<u>1</u> , 0
1 plays <i>paper</i>	<u>1</u> , 0	0, <u>1</u>	0, 0

- There is no cell with two underlined numbers $\Rightarrow$ New insight: There are games that have no Nash equilibrium!

(Micro III teaser: There is actually an equilibrium, but it requires looking beyond *pure strategies* ...)



The anatomy of games with Nash equilibrium

Technology and preferences	Behavior and Equilibrium
<i>Exogenous func./var./relationships:</i> $u_1(s_1, s_2)$ $u_2(s_2, s_1)$ S_1, S_2 <i>Endogenous variables:</i> s_1, s_2	<i>Agents' decisions:</i> Utility max. given opponent's strategy $\max_{s_1} u_1(s_1, s_2)$ $\max_{s_2} u_2(s_2, s_1)$ ↪ <i>Conditional behavior:</i> Best responses $s_1^*(s_2), s_2^*(s_1)$ <i>Equilibrium Conditions:</i> NE: Best responses to each other $s_1^*(\bar{s}_2) = \bar{s}_1 \text{ and } s_2^*(\bar{s}_1) = \bar{s}_2$



Dominating and dominated strategies I

Dominating and dominated strategies

A strategy for player 1, $s_1 \in S_1$, is *weakly dominated* by another strategy $s'_1 \in S_1$ if s'_1 makes player 1 weakly better off regardless what player 2 does

$$u_1(s'_1, s_2) \geq u_1(s_1, s_2) \quad \text{for all } s_2 \in S_2$$

s_1 is *strictly dominated* by s'_1 if s'_1 makes player 1 strictly better off regardless what player 2 does:

$$u_1(s'_1, s_2) > u_1(s_1, s_2) \quad \text{for all } s_2 \in S_2$$

A strategy for player 1, $s'_1 \in S_1$, is weakly / strictly *dominant* if it weakly / strictly dominates all other strategies $s_1 \in S_1 : s_1 \neq s'_1$



Dominating and dominated strategies II

- A strategy that is strictly dominated will never be played in Nash equilibrium, as it can never be a best response.
- A strictly dominating strategy will always be played in equilibrium: it will always be the only best response.
- This does NOT apply to weakly dominating / dominated strategies:
 - Weak dominance occurs if there is something the opponent could do that would make you indifferent between this strategy and another strategy ...
 - ... and it is possible that the opponent ends up doing exactly this in equilibrium.



Dominance in the Prisoner's Dilemma

	2 plays C	2 plays D
1 plays C	3, 4	-3, <u>6</u>
1 plays D	<u>5</u> , -2	<u>0</u> , <u>0</u>

- D is a strictly dominant strategy for both players (and C is strictly dominated)
- Can be easily seen because D is the unique best response regardless of which column / row the opponent plays



Dominance in the car game

	2 plays <i>left</i>	2 plays <i>right</i>
1 plays <i>left</i>	<u>0</u> , <u>0</u>	-1, -1
1 plays <i>right</i>	-1, -1	<u>0</u> , <u>0</u>

- The car game has no dominating or dominated strategies.
- Both sides of the road may be optimal depending on the opponent's choice.



A game of weak dominance

	2 plays A	2 plays B
1 plays A	<u>5</u> , 3	<u>5</u> , <u>4</u>
1 plays B	3, 1	<u>5</u> , <u>5</u>

- In this game, B is weakly dominated by A for player 1 (A gives utility 5 no matter what)
- But there is a Nash equilibrium where B is played: $(s_1, s_2) = (B, B)$



Socratic Quiz question

Which of the following is a weakly dominant strategy in the following game?

	2 plays O	2 plays F
1 plays O	3, 2	1, 2
1 plays F	0, 0	2, 3

- a) Strategy O for player 1.
- b) Strategy F for player 1.
- c) Strategy O for player 2.
- d) Strategy F for player 2.
- e) Don't know.



Back to Nørrebrogade

- Jeppe (1) and Johannes (2) try to meet at Nørrebrogade, but have to choose where they will each go; continuous strategies:

$$S_1 = S_2 = [0, 1000]$$

- New assumption: Jeppe (1) lives inside Copenhagen so he would like to meet further inside the city, Johannes (2) prefers to meet further outside. α_1 and α_2 indicate the strength of these preferences:

$$u_1(s_1, s_2) = -(s_1 - s_2)^2 - \alpha_1(s_1)^2$$

$$u_2(s_2, s_1) = -(s_1 - s_2)^2 - \alpha_2(s_2 - 1000)^2$$



Best response, player 1

- The strategies are now continuous variables so the matrix method does not work; instead utility maximization given s_2

$$\max_{s_1} u_1(s_1, s_2) = -(s_1 - s_2)^2 - \alpha_1 (s_1)^2$$

- Strictly concave utility, first-order condition:

$$-2(s_1 - s_2) - \alpha_1 \cdot 2 \cdot s_1 = 0 \iff s_1 = \frac{1}{1 + \alpha_1} s_2$$

- Best response function

$$s_1^*(s_2) = \frac{1}{1 + \alpha_1} s_2$$



Best response, player 2

- Similarly for player 2

$$\max_{s_2} u_2(s_2, s_1) = -(s_1 - s_2)^2 - \alpha_2(s_2 - 1000)^2$$

- Strictly concave utility, first order condition:

$$-2(s_1 - s_2) \cdot (-1) - \alpha_2 \cdot 2 \cdot (s_2 - 1000) = 0 \iff s_2 = \frac{1}{1 + \alpha_2}(s_1 + 1000\alpha_2)$$

- Best response function

$$s_2^*(s_1) = \frac{1}{1 + \alpha_2}(s_1 + 1000\alpha_2)$$



Nash equilibrium I

- In Nash equilibrium $(\bar{s}_1, \bar{s}_2)$ both players play best responses to each other so we have:

$$s_1^*(\bar{s}_2) = \bar{s}_1$$

$$s_2^*(\bar{s}_1) = \bar{s}_2$$

- We found the best response functions on the previous slide:

$$\frac{1}{1 + \alpha_1} \bar{s}_2 = \bar{s}_1 \quad (1)$$

$$\frac{1}{1 + \alpha_2} (\bar{s}_1 + 1000\alpha_2) = \bar{s}_2 \quad (2)$$



Nash equilibrium II

- Insert from equation (1) into equation (2):

$$\frac{1}{1+\alpha_2} \left(\frac{1}{1+\alpha_1} \bar{s}_2 + 1000\alpha_2 \right) = \bar{s}_2 \iff$$

$$\bar{s}_2 = \frac{1}{1+\alpha_2} \left(\frac{1}{1+\alpha_1} \bar{s}_2 \right) + \frac{1000\alpha_2}{1+\alpha_2} \iff$$

$$\bar{s}_2 = \frac{1}{(1+\alpha_2)(1+\alpha_1)} \bar{s}_2 + \frac{1000\alpha_2}{1+\alpha_2} \iff$$

$$\left(\frac{(1+\alpha_2)(1+\alpha_1) - 1}{(1+\alpha_2)(1+\alpha_1)} \right) \bar{s}_2 = \frac{1000\alpha_2}{1+\alpha_2} \iff$$

$$\bar{s}_2 = \frac{(1+\alpha_1)\alpha_2}{(1+\alpha_2)(1+\alpha_1) - 1} 1000$$



Nash equilibrium III

- Put back into equation (1):

$$\frac{1}{1+\alpha_1} \frac{(1+\alpha_1)\alpha_2}{(1+\alpha_2)(1+\alpha_1)-1} 1000 = \bar{s}_1 \iff$$
$$\bar{s}_1 = \frac{\alpha_2}{(1+\alpha_2)(1+\alpha_1)-1} 1000 \iff$$

- Nash equilibrium is:

$$\bar{s}_1 = \frac{\alpha_2}{(1+\alpha_2)(1+\alpha_1)-1} 1000$$
$$\bar{s}_2 = \frac{(1+\alpha_1)\alpha_2}{(1+\alpha_2)(1+\alpha_1)-1} 1000$$



Nash equilibrium, examples and comparative statics

- $\alpha_1 = \alpha_2 = 0.1$ (Small home bias) Johannes stands 524 meters down on Nørrebrogade, Jeppe 476 meters.
- $\alpha_1 = \alpha_2 = 0.3$ (Home bias of both is higher) indicate that Johannes stands 565 meters down on Nørrebrogade, Jeppe 435 meters.
- $\alpha_1 = 0.1, \alpha_2 = 0.3$ (Johannes has stronger home bias) gives Johannes standing 767 meters down on Nørrebrogade, Jeppe 698 meters.
- $\alpha_1 = 0.1, \alpha_2 = 1$ (Johannes has extreme home bias) gives Johannes standing 917 meters down on Nørrebrogade, Jeppe 833 meters.



Socratic Quiz Question

In the previous example, what would happen if Jeppe (player 1) were to move first, and only afterwards – when Jeppe has already made his decision — Johannes (player 2) is allowed to move?

- a) They would meet further outside the city.
- b) They would meet further inside the city.
- c) Nothing would change.
- d) There would be multiple equilibria.
- e) Don't know.



Games with many players

- Expanding to N players is conceptually simple, but notation can get a bit messy
- There will now be strategies for all players $s_1 \in S_1, s_2 \in S_2, \dots, s_N \in S_N$
- Useful notation: Let s_{-i} be a vector that contains the strategies of everyone other than player i , i.e.:
 - $s_{-1} = (s_2, s_3, \dots, s_N)$
 - $s_{-2} = (s_1, s_3, \dots, s_N)$
 - $s_{-i} = (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_N)$



Games with many players

Games with N players, definition

A game with N players, $i = 1, 2, \dots, N$, consists of:

- A strategy set of possible strategies for each player, $S_1, S_2, \dots, S_N$
- Utility functions that indicate each player's utility as a function of the strategies selected, $u_i(s_i, s_{-i})$, thus:

$$u_i : S_1 \times S_2 \times \dots \times S_N \rightarrow \mathcal{R}$$



Best response functions, many players

Best response functions, many players

A best response function for player i , $s_i^*(s_{-i})$, indicates an optimal strategy for player i as a function of other players' strategies

$$s_i^* : S_1 \times S_2 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_N \rightarrow S_i$$

and $s_i^*(s_{-i})$ solves the problem

$$\max_{s_i} u_i(s_i, s_{-i})$$



Nash equilibrium, many players

Nash equilibrium

A Nash equilibrium is a set of strategies for all players

$(\bar{s}_1, \bar{s}_2, \dots, \bar{s}_N) \in S_1 \times S_2 \times \dots \times S_N$ which are best responses to each other:

$$s_i^*(s_{-i}^-) = \bar{s}_i \quad \text{for all } i$$

It follows that in a Nash equilibrium, each player has maximized utility given what everyone else does.



Dominating and dominated strategies, many players

Dominating and dominated strategies, many players

A strategy for player i , $s_i \in S_i$, is *weakly dominated* by another strategy $s'_i \in S_i$ if s'_i makes player i weakly better off no matter what the other players do

$$u_i(s'_i, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_1 \times S_2 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_N$$

s_i is *strictly dominated* by s'_i if s'_i makes player i strictly better off no matter what the other players do:

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_1 \times S_2 \times \dots \times S_{i-1} \times S_{i+1} \times \dots \times S_N$$

A strategy for player i , $s'_i \in S_i$ is *weakly/strictly dominating* if it weakly/strictly *dominates* all other strategies $s_i \in S_i : s_i \neq s'_i$



What have we learned?

- What are simultaneous games with perfect information (and pure strategies)
- What are best response functions, Nash equilibria, dominating and dominated strategies
- Finding the best response and Nash equilibrium in such games, both with continuous and with discrete strategies

