Mikro II, lecture 11a

Preference relations and social preferences

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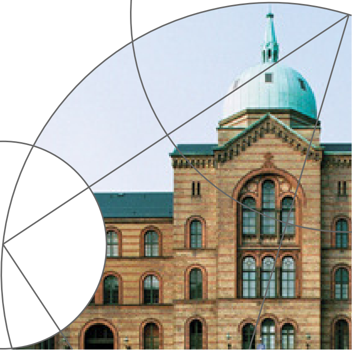


Plan for the lecture

1. Recap on preference relations
2. Intro on social preferences and social choice functions
3. Social preferences through democracy
4. Arrow's impossibility theorem

Agenda

- Our approach so far:
  1. Agents that explicitly maximized their utility (preferences) in the model ...
  2. ... and an (exogenous) economist (us) / ”the public” / society who looked at the model and discussed various policy actions and interventions that could be attractive ...
  3. ... where we have primarily assessed attractiveness by efficiency (Pareto Optimality)
- Now we will explore issues 2 and 3 in more detail.

Social Preferences

- We will focus on two issues in particular:
  - Here: In a society where people have different preferences, how do we (or how can we) make social decisions? (descriptive)
    - How does society decide about tax levels, public good provision, environmental regulation, etc.?
    - Simple answer: Democracy! BUT...
  - Here and in the next slideshow: Can we set some criteria for what society should base their decisions on? (more normative)
    - We have implicitly focused on efficiency, but often we have several efficient states, and what about distribution / equality?

Reminder, preference relations

preference relations, definition

Let A be a set of options. A preference relation $\succsim$ over A indicates which options are preferred over others.

- If for a couple of options $(x, y) \in A^2$, $x \succsim y$ applies, then we say that x is (weakly) preferred to y
- Let $\mathcal{P}_A$ be the sum of all preference relations of A (i.e. $\succsim \in \mathcal{P}_A$)

- Only new thing here is (maybe) the amount of $\mathcal{P}_A$: it just consists of all the preference relations we can imagine

  (Remember: besides $\succsim$ we can also talk about indifference, $\sim$, and strictly preferred, $\succ$)

Preference relations, a little more formal

Preference relations, definition

Let A be a set of options. A preference relation $\succsim$ over A **is a quantity of ordered pairs from** A, indicating which options are preferred over others **(that is.** $\succsim \subseteq A^2$**)**

- If for a couple of options $(x, y) \in A^2$ **it applies that** $(x,y) \in \succsim$ we say that x is (weakly) preferred to y and write $x \succsim y$
- Let $\mathcal{P}_A$ be the set of all preference relations on A (that is. $\succsim \in \mathcal{P}_A$)

- A preference relation can be considered as a long list indicating what is preferred over what

Examples of preference relations

- Let A consist of three options $A = \{x, y, z\}$

- We can define an example of a preference relation $\succsim$ as:

$$x \succsim y$$
$$y \succsim z$$

- Or written as a quantity (list):

$$\succsim = \{(x, y), (y, z)\}$$

Example: Edgeworth-economy I

Technology and Preferences	**Behavior and Equilibrium**
Exogenous func./var./relations: $u_A(x_1^A, x_2^A) = (x_1^A)^{\alpha} (x_2^A)^{1-\alpha}$ $u_B(x_1^B, x_2^B) = (x_1^B)^{\beta} (x_2^B)^{1-\beta}$ $\alpha, \beta, e_1^A, e_2^A, e_1^B, e_2^B$ The allocation should be possible *Endogenous variables:* $(x_1^A, x_2^A), (x_1^B, x_2^B)$	*The decisions of the agents:* $\hookrightarrow$ *Conditional behavior:* *Equilibrium Conditions:*

Example: Edgeworth-economy II

- Let A be all possible states in the economy, (very) formally:

$$A = \left\{ (x_1^A, x_2^A, x_1^B, x_2^B) \geq 0 \,|\, e_1^A + e_1^B = x_1^A + x_1^B \text{ and } e_2^A + e_2^B = x_2^A + x_2^B \right\}$$

- Define a preference relation (for A), $\succsim_A$, in which a state is preferred over another state if and only if it provides a (slightly) greater benefit for consumer A:

$$(x_1^A, x_2^A, x_1^B, x_2^B) \succsim_A (\bar{x}_1^A, \bar{x}_2^A, \bar{x}_1^B, \bar{x}_2^B) \iff u_A(x_1^A, x_2^A) \geq u_A(\bar{x}_1^A, \bar{x}_2^A)$$

- Utility functions are our standard way of thinking about agent preferences in relation to the possible states

Reminder: Rational preferences

Rational preferences (total pre-order), definition

Let A be a set of options and $\succsim$ be a preference relation for A.

- We say that $\succsim$ is *total* if for all $(x, y) \in A^2$ either $x \succsim y$ or $y \succsim x$ applies
- We say that $\succsim$ is *transitive* if for all $(x, y, z) \in A^3$ where $x \succsim y$ and $y \succsim z$ we have $x \succsim z$
- We say that $\succsim$ is *rational* (is a total pre-order) if $\succsim$ is total and transitive

- Total: All pairs can be compared; Transitive: If apples are better than pears and pears are better than oranges then apples are better than oranges.

Our first example I

- Our first preference relation example:

$$x \succsim y$$
$$y \succsim z$$

- This one is not total because we cannot compare x and z nor can we compare any of the options with themselves!

- If we need to change $\succsim$ to make it total we can start by adding:

$$x \succsim x$$
$$y \succsim y$$
$$z \succsim z$$

Our first example II

- Finally, if it is to be made total we must also be able to compare x and z; one option is to add:

$$z \succsim x$$

- If we do that, $\succsim$ is total. But in that case it is not transitive because $x \succsim y$ and $y \succsim z$ without $x \succsim z$

- If you add the opposite, $\succsim$ becomes both total and transitive and therefore rational (check yourself):

$$x \succsim z$$

(Overall, we have: $\succsim = \{(x,x),(y,y),(z,z),(x,y),(y,z),(x,z)\}$)

Preferences via utility functions

- Any preference relation defined via a utility function (such as $\succsim_A$ from earlier) is always total and transitive (i.e. rational)
- (One of the reasons economists care about whether preferences are rational is that it is a condition for being able to use utility functions)

Socrative Quiz Question

True or false: Any preference relation that is total and transitive can be described by a utility function.

Aggregation of preferences

- We now want to try to look at a society where agents have different preferences and can make decisions together
- In other words, we would like to *aggregate* individual preferences to obtain the society preference
- The way we do it is called a social choice function

Social choice functions

Social choice functions

Consider an economy with N agents that we index with i (N is odd). Let A be a number of possible states of the economy:

- Each agent is equipped with a preference relation over A. We denote agent i's preference relation $\succsim_i$ and let R denote total preferences in the economy, i.e.

$$R = (\succsim_1, \succsim_2, \ldots, \succsim_N) \in \mathcal{P}_A$$

- A social choice function $f : \mathcal{P}_A^N \to \mathcal{P}_A$ is a function that aggregates the individual preferences into a single social preference, $\succsim^*$, that is:

$$\succsim^* = f(R) = f(\succsim_1, \succsim_2, \ldots, \succsim_N)$$

Social choice functions, discussion

- The social choice function (SCF) is a rule (or system) that determines how we make decisions (e.g. democracy)
- If we put a collection of preferences (a collection of agents) into the SCF it will tell us what the society will decide jointly
- A little more accurately, the SCF spits out a social preference that tells us how society ranks the various possible states

Example: Dictatorship

- One way to make society's decisions is to make an agent a dictator, e.g. agent 1

- The *Dictator* SCF can be expressed as:

$$f(\succsim_1, \succsim_2, ..., \succsim_N) = \succsim_1$$

or equivalently as $\succsim^*$, thus:

$$x \succsim^* y \iff x \succsim_1 y$$

Example: Democracy

- Another way to make decisions is *Democracy*
- Society must choose between x (high tax?) and y (low tax?) by simple democracy: Agents just vote between the two options
- Written mathematically, the *Democracy* SCF corresponds to:

$$x \succsim^* y \quad \Longleftrightarrow \quad \text{it applies that } x \succsim_i y \text{ for a majority of the agents}$$

Socrative Quiz Question

True or false: If the preferences of all agents in an economy are rational, then so will be the corresponding Democracy SCF.

Analysis of social choice functions

- We will now examine various Social Choice Functions f in practice; How do they work? What characteristics do they have?

- The final social preferences $\succsim^*$ depend on which SCF, f, we have chosen and which preferences, R, the agents have
  - We will (basically) allow R to be anything, $R \in \mathcal{P}_A$ and investigate the consequences of different f
  - Intuition: We will investigate whether our way of making society's decisions is ”good”, regardless of what preferences (what types of people) might end up living in the society
  - (However, we typically assume that agents have rational preferences)

Democracy and decision I

- Let's look at the democracy SCF: Society compares opportunities via polls

$$x \succsim^* y \quad \Longleftrightarrow \quad \text{it applies that } x \succsim_i y \text{ for a majority of the agents}$$

- Basically, this seems like a really good SCF

- It will be helpful here to think about how the resulting social preference is translated into society choosing an option

- Social utility maximization: society chooses something that is preferred over everything else; mathematically we say that x^* is optimal when:

$$x^* \succsim^* y \text{ for all } y \in A$$

Democracy and decision II

- One obvious way to find such an optimal decision under democracy is:
  1. Choose two options x and y and hold vote
  2. The winner of the voting goes on to a vote against another option z which it has not yet won over
  3. Step 2 is repeated until there is an option that has won over all the other options

- This decision-making process is natural:
  - Just corresponds to sequentially comparing two options as prescribed under the SCF
  - Easy to show mathematically that the process will end up choosing an optimal option x^* (if such an option exists)

Democracy, example 1

- Let $A = \{x, y, z\}$ and $N = 3$ and let (mathematically imprecise) the three agents have rational preferences described as follows:

$$\begin{array}{lccccc} \text{Agent 1:} & x & \succsim_1 & z & \succsim_1 & y \\ \text{Agent 2:} & y & \succsim_2 & z & \succsim_2 & x \\ \text{Agent 3:} & z & \succsim_3 & x & \succsim_3 & y \end{array}$$

- Now we hold polls starting with x against y:
  1. x vs y; voting: $x, y, x \Rightarrow x$ wins
  2. x vs z; voting: $x, z, z \Rightarrow z$ wins
  3. z vs y; voting: $z, y, z \Rightarrow z$ wins
  4. z has beaten all other options, so z wins

Democracy, example 2

- Now look at these (rational) preferences instead:

$$\begin{array}{llllll} \text{Agent 1:} & x & \succsim_1 & y & \succsim_1 & z \\ \text{Agent 2:} & y & \succsim_2 & z & \succsim_2 & x \\ \text{Agent 3:} & z & \succsim_3 & x & \succsim_3 & y \end{array}$$

- Now we hold polls starting with x against y:
  1. x vs y; voting: $x, y, x \Rightarrow x$ wins
  2. x vs z; voting: $x, z, z \Rightarrow z$ wins
  3. z vs y; voting: $y, y, z \Rightarrow y$ wins
  4. y vs x; voting: x, y, x ...
  5. LOOP!

Condorcet cycles

- The example shows that under democracy situations can occur where one can keep voting without finding a winner, so-called *Condorcet cycles*

- Mathematically, the problem is that the social preference ends up not being transitive, for example we have:

$$x \succsim^* y \text{ and } y \succsim^* z$$

  but it does not hold that:

$$x \succsim^* z$$

- Because social preferences are not transitive, there is no longer an optimal option x^* that beats all the others (a *Condorcet* winner)

Democracy is problematic

- Note that the problem of intransitive social preferences arose even though all agents' preferences *are* transitive
- The democracy SCF thus has the unfortunate characteristic that it can result in intransitive social preferences (and Condorcet cycles)
- Can we come up with a way to fix the democratic process?
  - One option: Put (other) restrictions on how to vote?
  - Simple (realistic) bid: Do not allow voting again for an option that has already lost

This change removes the cycles ...

- Democracy, example:

$$\begin{array}{lccccc} \text{Agent 1:} & x & \succsim_1 & y & \succsim_1 & z \\ \text{Agent 2:} & y & \succsim_2 & z & \succsim_2 & x \\ \text{Agent 3:} & z & \succsim_3 & x & \succsim_3 & y \end{array}$$

- Now we are holding polls starting with x against y:
  1. x vs y; voting: $x, y, x \Rightarrow x$ wins
  2. x vs z; voting: $x, z, z \Rightarrow z$ wins
  3. Both x and y have lost a vote, so z wins

... but now the order of voting means a lot

- Democracy, example 2:

$$\begin{array}{lccccc} \text{Agent 1:} & x & \succsim_1 & y & \succsim_1 & z \\ \text{Agent 2:} & y & \succsim_2 & z & \succsim_2 & x \\ \text{Agent 3:} & z & \succsim_3 & x & \succsim_3 & y \end{array}$$

- Now we are holding polls starting **with x against z**:
  1. x vs z; voting: $x, z, z \Rightarrow z$ wins
  2. z vs y; voting: $y, y, z \Rightarrow y$ wins
  3. Both x and z have lost a vote, **hence y wins**

Agenda setting

- If we impose restrictions on the voting process, the order determines the overall winner
- Voting results can thus be manipulated by changing the voting agenda.
- Known as *Agenda setting power*.
- Similar phenomena apply to other changes to the voting rules.
- Example: Vote for more options than two? Now the outcome is influenced by what other options are included in the vote.

Another “solution”

- Another (important) way to get rid of our anti-democratic conclusions is to impose multiple assumptions on preferences

Single-peaked Preferences, definition

Let $A \subseteq \mathcal{R}$. We say that agents have *single-peaked* preferences if for each agent i there exists an *ideal point* $x_i \in A$ such that for all $(y, z) \in A^2$ we have

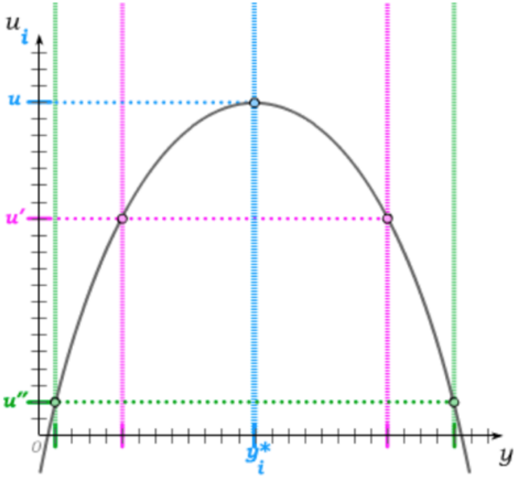
$$x_i \geq y \geq z \Rightarrow y \succsim_i z$$

and

$$x_i \leq y \leq z \Rightarrow y \succsim_i z$$

- Intuition: Every agent has a favorite option x_i and just wants to get as close to this point as possible

Graphical example



- Let agent i have preferences described by the utility function:

$$u_i(x) = -(x_i - x)^2$$

- Utility is highest when $x = x_i$
- Otherwise, it's just about being as close as possible

Socrative Quiz question

True or false: Preferences described by the below utility function are not single-peaked.

$$u_i(x) = -(x_i - x)^2 - (x_i - x)^4$$

Median Voter Theorem

- If the preferences are single-peaked, the median voter theorem tells us that we are getting rid of the unfortunate previous conclusions:

Median Voter Theorem

Assume that agents have single-peaked preferences and let x_i indicate the ideal point of agent i. We say that agent i is a median voter if his ideal point is equal to the median of all the ideal points:

$$x_i = med\, x_j$$

It is now true that the ideal point of the median voter is socially optimal (is a Condorcet winner) under the Democracy SCF:

$$med\, x_j \succsim^* y \quad \text{for all} \quad y \in A$$

Median Voter Theorem, proof (sketch)

- Let y be any option in A, it applies that:
  - If $y \leq med\, x_j$, then $med\, x_j$ is preferred by the median voter and anyone with ideal point above the median $\Rightarrow$ majority
  - If $y \geq med\, x_j$ is $med\, x_j$ preferred by the median voter and anyone with ideal point below the median $\Rightarrow$ majority
- The median voter's ideal point wins every pair vote and is socially optimal (is a Condorcet winner)

Median Voter Theorem, discussion

- The median voter theorem provides some conditions on the preferences that remove the problem of Condorcet cycles
- The median voter theorem also provides a descriptive prediction:
  - Under democracy we always end up choosing what the median voter wants
  - Formalizes the intuitive idea that politics always ends up doing something ”in the middle” of people’s preferences

Median voter theorem, disadvantages

- The result requires a one-dimensional policy with a clear ranking ($A \subseteq \mathcal{R}$):
  - Does not work with options without a ranking:
    $A = \{\text{paint red, paint white, paint grey}\}$
  - Does not work with two-dimensional policies:
    $$A = \{\text{high immigration and high taxes, high immigration and low taxes, low immigration and high taxes, low immigration and low taxes}\}$$

- Single-peaked may not hold even for one-dimensional policies with a clear ranking:
  - Eg. how much money are we going to spend on primary school:
    $A = \{\text{a lot, a little, none}\}$
  - If a poor public school gets someone to use private school we can get
    $\text{a lot} \succsim \text{none} \succsim \text{a little}$

Socrative Quiz question

True or false: A democracy with a set of voters with rational and single-peaked preferences is equivalent to a dictatorship where the median voter is the dictator.

Arrow's impossibility theorem

- We will end by looking at the famous Arrow's impossibility theorem
- A great economist: Kenneth Arrow, Nobel Prize 1972 (youngest until 2019)
- Arrow's famous analysis is based on:
  1. List of five conditions that a "good" SCF should fulfill
  2. Mathematically investigate which SCFs meet these conditions

I: Universal domain (UD)

Universal domain (UD)

The social choice function f must be defined for all rational preferences over A. That is:

$$f : \mathcal{Q}_A^N \to \mathcal{P}_A$$

where $\mathcal{Q}_A \subseteq \mathcal{P}_A$ is defined as the amount of rational preferences over A

- Intuition: Arrow wants his SCF to work no matter how the preferences look, as long as they are rational (total and transitive)

  (We have implicitly assumed this already)

II: The Pareto criterion (PU)

The Pareto criterion (PU)

For any pair of options $(x, y) \in A^2$ it should apply that if all agents prefer x over y then the social preferences should do so too:

$$x \succsim_i y \text{ for all } i \quad \Rightarrow \quad x \succsim^* y$$

for $\succsim^* = f(\succsim_1, \succsim_2, \ldots, \succsim_N) = f(R)$

- Intuition: Arrow wants that if all individuals like apples better than pears then society should also like apples better than pears

III: Rationality criterion (R)

Rationality criterion (R)

The social preferences that come out of the social choice function, $\succsim^* = f(R)$ must be rational, i.e. total and transitive

- Intuition part 1: Arrow wants society to be able to compare all possibilities (total)
- Intuition part 2: If society prefers apples over pears and pears over oranges, it should also prefer apples over oranges

IV: Independence of irrelevant alternatives (IIA)

Independence of irrelevant alternatives (IIA)

Let $R = (\succsim_1, \succsim_2, \dots \succsim_N) \in Q_A^N$ and $R' = (\succsim_1', \succsim_2', \dots \succsim_N') \in Q_A^N$ be two possible sets of preferences for the agents and let $\succsim^* = f(R)$ and $\succsim^{*\prime} = f(R')$ be the associated social preferences. For any pair of options $(x, y) \in A^2$, it should apply that if all agents rate x and y equally under R and R', then the social preferences must rank this way as well, i.e. if

$$x \succsim_i y \iff x \succsim_i' y$$

it should apply that

$$x \succsim^* y \iff x \succsim^{*\prime} y$$

IIA, intuition

- Consider a set of preferences for the agents $R = (\succsim_1, \succsim_2, \ldots \succsim_N)$ and look at two options x and y

- We now imagine that we change the preferences of one or more agents, but without changing their relative valuation of x and y, ie. we only change preferences regarding one (or more) irrelevant options z
The IIA condition now states that this should not affect whether society prefers x to y

"Agents' preferences for liquor should not affect whether the society prefers caramel to chocolate "

V: The non-dictatorship criterion (ND)

The non-dictatorship criterion (ND)

For all agents i, there must be at least one $R = (\succsim_1, \succsim_2, ..., \succsim_N) \in Q_A^N$ where i does not dictate social preferences, i.e. where:

$$f(R) \neq \succsim_i$$

- Intuition: Arrow doesn't want the SCF to just mean that one particular agent always determines the results
- Note: Nechyba uses a different (weaker) definition of dictatorship

Arrow’s Impossibility theorem

- Now we have reviewed the five (reasonable?) conditions that Arrow sets for a good SCF

- We will not spend time reviewing Arrow’s analysis (evidence), but just look at his (remarkable) conclusion:

Arrow's Impossibility theorem

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Arrow's Impossibility Theorem

If $N \geq 2$ and A contains at least 3 different options then there is *no* social choice function that satisfies the conditions (UD), (PU), (R), (IIA) and (ND).

- Wild result! There is no systematic way of aggregating preferences that meets Arrow's desired conditions

Dictatorship is ”possible”

- In a way, it gets even worse (if you are pro democracy):

The dictator function complies with the conditions

A social choice function that makes one of the agents a dictator meets the conditions (UD), (PU), (R) and (IIA)

- If we drop our dictator condition, dictatorship is a usable SCF

Arrow's Impossibility theorem, discussion

- The Impossibility Theorem emphasizes how difficult it is to aggregate preferences of (potentially) disagreeing agents
- The result can be seen as a severe blow to the idea that it is possible to arrive at a good / fair arrangement for society's preferences
- Arrow is quoted for:

  "I'm not saying that all decision-making systems are always bad, just that every decision-making system will sometimes work less well."

What now?

- If you want to go ahead with a project to find a good SCF for the society, then you have to relax one of Arrow's requirements
- There is a great deal of literature examining what happens if one or more of Arrow's criteria are compromised (see especially Amartya Sen's work)
- You can see our next topic as an example of this

What have we learned?

- What is a social choice function
- Democracy can provide intransitivity and Condorcet cycles
- What are single-peaked preferences and what do they mean
- What does Arrow’s impossibility theorem say?