Problem Set 5

Macroeconomics III

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Problem 1

The maximization problem of the individual is:

$$\max_{c_{1t},\ c_{2t+1}} \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

$$\text{s.t.} \quad c_{1t} = w_t - s_t$$

$$c_{2t+1} = (1 + r_{t+1} - \delta)s_t$$

Where the usual gross interest applies: $R_t = 1 + r_t - \delta$. The problem was solved in problem set 4 and implies:

$$s_t = \frac{\beta}{1+\beta} w_t$$

The production function is:

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}$$

Which implies:

$$r_t = \alpha k_t^{\alpha - 1} \qquad \text{and} \qquad w_t = (1-\alpha)k_t^{\alpha}$$

Note: We're not given info about population growth but:

$$L_{t+1} = (1+n)L_t$$

Problem 1a

Show that capital per unit of labour evolves according to

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)k_t^{\alpha}$$

We insert the wage rate into the optimal savings:

$$s_t = \frac{\beta}{1+\beta}w_t = \frac{\beta}{1+\beta}(1-\alpha)k_t^{\alpha}$$

The savings made in period t is equal to the capital in period $t+1$.

We follow the calculations from problem set 4 with $K_{t+1} = S_t$:

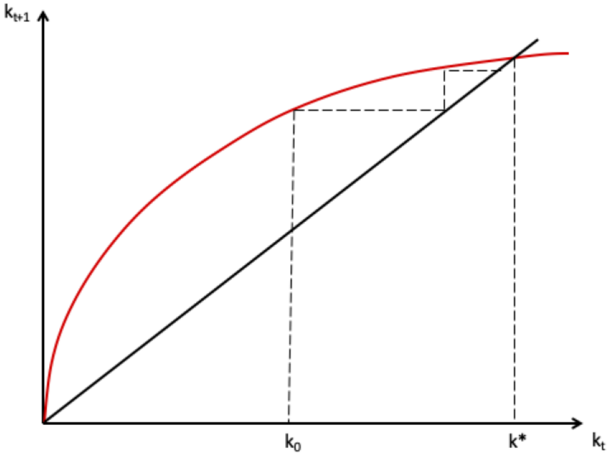
$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{S_t}{L_{t+1}} = \frac{L_t s_t}{(1+n)L_t} = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)k_t^{\alpha}$$

Strictly increasing and concave.

Problem 1a - Evolution of capital

Analyze graphically the evolution of the economy from $k_0 < k^*$ to SS.

$$k_{t+1} = \frac{1}{1+n} \frac{\beta}{1+\beta} (1-\alpha) k_t^{\alpha} \tag{1}$$



Problem 1a - Evolution of capital

Explain the problems to find and analytical solution that would arise without the assumption that instantaneous utility is logarithmic.

These results are based on log-utility, where Income effect $=$ substitution effect following a change in r_t.
$\rightarrow$ Savings rate is independent of r_t

Suppose we had CRRA utility with very risk-averse agents, $\sigma > 1$.

- Income effect $>$ substitution effect.
- A decrease in the interest rate will lead households to save more.
- Since households save more, the interest rate decreases further.
- Then they save even more and so forth.

The savings rate, therefore, depends on future capital level.

$$k_{t+1} = \frac{s_t(w_t, r_{t+1})}{1+n}$$

There can therefore be multiple equilibria.

Problem 1b - Equilibrium interest rate

Show that it cannot be ruled out that the steady state is dynamically inefficient, i.e. that $r^* < n + \delta$.

We find the steady state capital level:

$$k = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)k^{\alpha}$$

$$k^{1-\alpha} = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)$$

$$k^* = \left(\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right)^{\frac{1}{1-\alpha}}$$

Next, we find the steady state interest rate:

$$r^* = \alpha k^{*\alpha-1} = \alpha\left(\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right)^{\frac{\alpha-1}{1-\alpha}} = \frac{\alpha(1+n)(1+\beta)}{\beta(1-\alpha)}$$

We cannot rule out dynamic inefficiency.

$$r^* = \frac{\alpha(1+n)(1+\beta)}{\beta(1-\alpha)} \lesseqgtr n + \delta$$

Problem 1b - Dynamic inefficiency

In the case of dynamic inefficiency, the following holds:

$$r^* = \frac{\alpha(1+n)(1+\beta)}{\beta(1-\alpha)} < n + \delta$$

Meaning that the capital level is inefficiently high.

The Social planner could, therefore, make all generations better by lowering the capital level as it would increase consumption for all generations.

Problem 1b - Increasing welfare and FWT

Explain how in this situation, the utility of all individuals could be increased if k was decreased and explain why this is not in contradiction with the first welfare theorem.

Relation to the First Welfare Theorem: The FWT is not contradicted since there are infinitely many households so the conditions are not fulfilled.

This can be achieved by implementing a Pay-As-You-Go (PAYG) scheme: Young households pay a contribution d and receive a return of $(1+n)d$ when old.

Return of savings: $R^* = 1 + r^* - \delta$

Return of the PAYG scheme: $1 + n$

Dynamic inefficiency implies:

$$r^* < n + \delta \implies R^* < 1 + n$$

Hence, the households would be made better of by the PAYG if the economy exhibits dynamic inefficiency.

Problem 1c - Household problem

Households pay a fraction of τ of their income when young and receive benefits $b_{t+1} = \tau w_t(1 + r_{t+1})$ when old. The household problem is:

$$\max_{c_{1t},\ c_{2t+1}} \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

$$\text{s.t.} \quad c_{1t} = (1-\tau)w_t - s_t$$

$$c_{2t+1} = (1 + r_{t+1} - \delta)s_t + \tau w_t(1 + r_{t+1} - \delta)$$

Substituting in the budget constraints:

$$U = \ln\big((1-\tau)w_t - s_t\big) + \beta \ln\big((1 + r_{t+1} - \delta)s_t + \tau w_t(1 + r_{t+1} - \delta)\big)$$

Maximizing wrt. s_t yields:

$$\frac{\beta}{s_t + \tau w_t} = \frac{1}{(1-\tau)w_t - s_t}$$

$$s_t + \tau w_t = \beta\big((1-\tau)w_t - s_t\big)$$

$$s_t = \frac{\beta}{1+\beta} w_t - \tau w_t$$

Problem 1c - Savings rate - extra derivations

$$
\begin{aligned}
\frac{\beta}{s_t + \tau w_t} &= \frac{1}{(1-\tau)w_t - s_t} \\
s_t + \tau w_t &= \beta\big((1-\tau)w_t - s_t\big) \\
s_t &= \beta(1-\tau)w_t - \beta s_t - \tau w_t \\
s_t(1+\beta) &= \beta(1-\tau)w_t - \tau w_t \\
s_t &= \frac{\beta}{1+\beta}(1-\tau)w_t - \tau\frac{w_t}{1+\beta} \\
&= \frac{\beta}{1+\beta}w_t - \frac{\beta}{1+\beta}\tau w_t - \frac{1}{1+\beta}\tau w_t \\
&= \frac{\beta}{1+\beta}w_t - \frac{1}{1+\beta}\beta\tau w_t - \frac{1}{1+\beta}\tau w_t \\
&= \frac{\beta}{1+\beta}w_t - \frac{1}{1+\beta}(1+\beta)\tau w_t \\
&= \frac{\beta}{1+\beta}w_t - \tau w_t
\end{aligned}
$$

Problem 1c - Savings rate

Hence, the savings are:

$$s_t = \frac{\beta}{1+\beta} w_t - \tau w_t$$

Which implies that the savings rate is:

$$\frac{\beta}{1+\beta} - \tau$$

Does this make sense?
Yes. Suppose you have decided to save 100$ for the next year. If the government forces you to save 50$, then you simply choose to save 50$ yourself.

Hence, the government mandated savings do not affect your savings as long as they are below the optimal savings rate, $\tau < \frac{\beta}{1+\beta}$.

Problem 1c - Consumption and capital accumulation

Consumption when young is:

$$c_{1t} = (1-\tau)w_t - s_t = (1-\tau)w_t - \left(\frac{\beta}{1+\beta}w_t - \tau w_t\right) = \frac{1}{1+\beta}w_t$$

Consumption when old is:

$$c_{2t+1} = (1 + r_{t+1} - \delta)(s_t + \tau w_t) = (1 + r_{t+1} - \delta)\left(\frac{\beta}{1+\beta}w_t - \tau w_t + \tau w_t\right)$$
$$= (1 + r_{t+1} - \delta)\frac{\beta}{1+\beta}w_t$$

The aggregate amount of savings at time t is the sum om the individual savings and the "forced" savings. The capital accumulation is given by:

$$k_{t+1} = \frac{s_t + s_t^G}{1+n} = \frac{s_t + \tau w_t}{1+n} = \frac{1}{1+n}\left(\frac{\beta}{1+\beta}w_t - \tau w_t + \tau w_t\right)$$
$$= \frac{1}{1+n}\frac{\beta}{1+\beta}w_t = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)k_t^{\alpha}$$

The economy is unchanged.

Problem 1d - Public consumption instead of benefits

Assume instead that the revenue from the wage income tax is used to finance public consumption

The problem is now:

$$\max_{c_{1t},\ c_{2t+1}} \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

$$\text{s.t.} \quad c_{1t} = (1-\tau)w_t - s_t$$

$$c_{2t+1} = (1 + r_{t+1} - \delta)s_t$$

The results from PS4 apply but with wages as $\tilde{w}_t = (1-\tau)w_t$:

$$s_t = \frac{\beta}{1+\beta}(1-\tau)w_t = \frac{\beta}{1+\beta}(1-\tau)(1-\alpha)k_t^{\alpha}$$

$$k_{t+1} = \frac{1}{1+n}\frac{\beta}{1+\beta}(1-\alpha)(1-\tau)k_t^{\alpha}$$

Consumption when young is, therefore:

$$c_{1t} = (1-\tau)w_t - s_t = (1-\tau)w_t - \frac{\beta}{1+\beta}(1-\tau)w_t = \frac{1}{1+\beta}(1-\tau)w_t$$

Problem 1e - Effect of the tax

Analyze how the tax affects capital, the wage per unit of labour, the interest rate, consumption of the young and of the old from the period when the tax is introduced and afterwards.

New steady state level capital:

$$\tilde{k} = \left(\frac{\beta(1-\alpha)(1-\tau)}{(1+n)(1+\beta)}\right)^{\frac{1}{1-\alpha}} < \left(\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right)^{\frac{1}{1-\alpha}} = k^*$$

Capital level decreases, wages decrease and the interest rate increases.[1]

The old: Unchanged consumption on impact, since they pay no tax.

The young: Lifetime consumption:

$$C_t = c_{1t} + \frac{c_{2t+1}}{R_{t+1}} = (1-\tau)w_t - s_t + \frac{R_{t+1}s_t}{R_{t+1}} = (1-\tau)w_t \qquad (2)$$

Their lifetime consumption and thus welfare is lower. There are general equilibrium effects ($r_t \uparrow$ and $w_t \downarrow$) but the overall effect is negative.

[1]Note: this is a multi-period process.

Problem 2

We consider an OLG model with a PAYG scheme where individuals pay d when young and receive d when old. The maximization problem is given by:

$$\max_{c_{1t},\ c_{2t+1}} \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

$$\text{s.t.} \quad c_{1t} = w_t - s_t - d$$

$$c_{2t+1} = R_{t+1} s_t + d$$

Production for firm i is given by:

$$Y_t^i = A(K_t^i)^\alpha \left(K_t N_t^i\right)^{1-\alpha}$$

We normalize population such that $N_t = 1$.

Problem 2a - FOCs for the firm

Show that r_t is independent of the capital stock, and $w_t = (1-\alpha)AK_t$

Taking first order conditions yields:

$$\text{FOC wrt. } N_t: \qquad w_t = A(1-\alpha)(K_t^i)^\alpha K_t^{1-\alpha}(N_t^i)^{-\alpha} \tag{3}$$

$$\text{FOC wrt. } K_t: \qquad r_t = A\alpha(K_t^i)^{\alpha-1} K_t^{1-\alpha}(N_t^i)^{1-\alpha} \tag{4}$$

All firms are identical, so they choose the same relative capital and labour levels: $\frac{K_t^i}{N_t^i} = \frac{K_t}{N_t} = K_t$. We can therefore rewrite the FOCs:

$$w_t = A(1-\alpha)\Big(\frac{K_t^i}{N_t^i}\Big)^\alpha K_t^{1-\alpha} = A(1-\alpha)K_t^\alpha K_t^{1-\alpha} = A(1-\alpha)K_t$$

$$r_t = A\alpha\Big(\frac{K_t^i}{N_t^i}\Big)^{\alpha-1} K_t^{1-\alpha} = A\alpha K_t^{\alpha-1}K_t^{1-\alpha} = A\alpha$$

Problem 2b - Savings

Find saving and capital accumulation for the steady state

By inserting the budget constraints into the utility function, the problem becomes maximizing the following wrt. s_t:

$$U_t = \ln(w_t - s_t - d) + \beta \ln(R_{t+1} s_t + d)$$

The FOC is then:

$$\begin{aligned}
s_t R_{t+1} + d &= \beta R_{t+1}(w_t - s_t - d) \\
s_t R_{t+1}(1+\beta) &= \beta R_{t+1}(w_t - d) - d \\
s_t &= \frac{\beta}{1+\beta}(w_t - d) - \frac{1}{R_{t+1}} d \\
s_t &= \frac{\beta}{1+\beta}\left(w_t - d - \frac{1}{\beta R_{t+1}} d\right)
\end{aligned}$$

Note: The economy is efficient since $r > n = 0$. The PAYG scheme is therefore inefficient and decreases the consumption. The PAYG crowds out private savings.

Problem 2b - Capital accumulation

Capital accumulation is given by:

$$k_{t+1} = s_t = \frac{\beta}{1+\beta}\big(w_t - d - \frac{1}{\beta R_{t+1}} d\big)$$
$$= \frac{\beta}{1+\beta}\Big(A(1-\alpha)k_t - d\big(1 + \frac{1}{\beta(1+A\alpha)}\big)\Big)$$

Hence, capital accumulation is linear. There is a possible steady state.

Is it stable?

If the slope is greater than one, then the model features endogenous growth. If the slope is less than one, the model breaks down.

Problem 2c - Eliminating the PAYG scheme

At time t_0 the PAYG scheme is eliminated.

- The young in period t_0 pay γd.
- The young in all following periods pay $r(1-\gamma)d$ to service the debt.
- The old in period t_0 still receive d.
- The old in all following periods receive nothing.

The problem for young individuals in period t_0:

$$\max_{c_{1t_0},\, c_{2t_0+1}} \ln(c_{1t_0}) + \beta \ln(c_{2t_0+1})$$

$$\text{s.t.} \quad c_{1t_0} = w_{t_0} - \gamma d - s_{t_0} = \tilde{W}_{t_0} - s_{t_0}$$

$$c_{2t_0+1} = (1+r)s_{t_0}$$

The problem for young individuals in period $t > t_0$:

$$\max_{c_{1t},\, c_{2t+1}} \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

$$\text{s.t.} \quad c_{1t} = w_t - r(1-\gamma)d - s_t = \bar{W}_t - s_t$$

$$c_{2t+1} = (1+r)s_t$$

Problem 2 - Period t_0 savings for $\gamma \in [0, 1]$

The problem for young individuals in period t_0:

$$\max_{c_{1t_0},\ c_{2t_0+1}} \ln(c_{1t_0}) + \beta \ln(c_{2t_0+1})$$

$$\text{s.t.} \quad c_{1t_0} = w_{t_0} - \gamma d - s_{t_0} = \tilde{W}_{t_0} - s_{t_0}$$

$$c_{2t_0+1} = (1+r) s_{t_0}$$

Optimal savings imply:

$$s_{t_0} = \frac{\beta}{1+\beta} \tilde{W}_{t_0} = \frac{\beta}{1+\beta} (w_{t_0} - \gamma d)$$

Savings increase compared to the PAYG for any γ. Agents know that they receive nothing when old and their disp. income is now greater for $\gamma < 1$.

The government issues debt, which changes the capital accumulation

$$k_{t+1} + (1-\gamma) d = s_t \quad \Longrightarrow \quad k_{t+1} = s_t - (1-\gamma) d$$

Hence, the government debt lowers capital accumulation.

- Savings increase, which increases capital.
- Agg. savings also cover government debt, which lowers capital.

Problem 2 - Period t_0 savings for $\gamma = 0$

The savings and capital accumulation become:

$$s_{t_0} = \frac{\beta}{1+\beta} w_{t_0}$$

$$k_{t_0+1} = s_{t_0} - d = \frac{\beta}{1+\beta} w_{t_0} - d$$

Hence, the savings rate increases. Remember the capital accumulation for the PAYG system:

$$k_{t+1} = \frac{\beta}{1+\beta}\Big(A(1-\alpha)K_t - d\big(1 + \frac{1}{\beta(1+A\alpha)}\big)\Big)$$

The capital accumulation is lower for:

$$1 > \frac{\beta}{1+\beta}(1 + \frac{1}{\beta R}) \implies 1 > \frac{1}{R}$$

Which is the case for this economy. For $\gamma = 0$, savings increase but capital accumulation decrease.

Problem 2 - Period t_0 savings for $\gamma = 1$

The savings and capital accumulation become:

$$s_{t_0} = \frac{\beta}{1+\beta}(w_t - d)$$
$$k_{t_0+1} = s_{t_0} = \frac{\beta}{1+\beta}(w_t - d)$$

Hence, the savings rate increases. Remember the capital accumulation for the PAYG system:

$$k_{t+1} = \frac{\beta}{1+\beta}\Big(A(1-\alpha)K_t - d\big(1 + \frac{1}{\beta(1+A\alpha)}\big)\Big)$$

The capital accumulation is higher for:

$$\frac{\beta}{1+\beta} < \frac{\beta}{1+\beta}(1 + \frac{1}{\beta R})$$

Which holds. Hence, both savings and capital accumulation is higher for $\gamma = 1$.

Problem 2 - Welfare effects in period t_0 for $\gamma \in [0, 1]$

To evaluate whether there is political support, we consider the welfare effects for young and old. To do so, it is sufficient to look at the lifetime budget.

The old:
The policy change does not affect their maximization problem so they are indifferent.

The young:
The lifetime budget constraints are given by:

$$\text{With PAYG:} \quad c_{1t} + \frac{c_{2t+1}}{1+r} = w_t - d + \frac{d}{1+r} = w_t - \frac{r}{1+r}d$$

$$\text{Without PAYG:} \quad c_{1t} + \frac{c_{2t+1}}{1+r} = w_t - \gamma d$$

The current young will support dismantling the system for $\gamma < \frac{r}{1+r}$.

$\gamma = 0 \rightarrow$ The current young are better off.

$\gamma = 1 \rightarrow$ The current young are worse off.

Problem 2 - Period $t > t_0$ savings for $\gamma \in [0, 1]$

The problem for young individuals in period $t > t_0$:

$$\max_{c_{1t},\ c_{2t+1}} \ln(c_{1t}) + \beta \ln(c_{2t+1})$$

$$\text{s.t.} \quad c_{1t} = w_t - r(1-\gamma)d - s_t = \bar{W}_t - s_t$$

$$c_{2t+1} = (1+r)s_t$$

Optimal savings imply:

$$s_t = \frac{\beta}{1+\beta}\bar{W}_t = \frac{\beta}{1+\beta}(w_t - r(1-\gamma)d)$$

With the capital accumulation:

$$k_{t+1} = s_t - (1-\gamma)d = \frac{\beta}{1+\beta}[w_t - r(1-\gamma)d] - (1-\gamma)d$$

For the two cases:

$$\gamma = 0 \qquad k_{t+1} = s_t - (1-\gamma)d = \frac{\beta}{1+\beta}[w_t - rd] - d$$

$$\gamma = 1 \qquad k_{t+1} = s_t - (1-\gamma)d = \frac{\beta}{1+\beta}w_t$$

Problem 2 - Welfare effects in period $t > t_0$ for $\gamma \in [0, 1]$

We consider the lifetime budgets for the future young:

$$\text{With PAYG:} \qquad c_{1t} + \frac{c_{2t+1}}{1+r} = w_t - \frac{r}{1+r}d$$

$$\text{Without PAYG:} \qquad c_{1t} + \frac{c_{2t+1}}{1+r} = w_t - r(1-\gamma)d$$

The future young will benefit from dismantling the PAYG system if:

$$\frac{1}{1+r} > (1-\gamma) \implies \gamma > \frac{r}{1+r}$$

Note: this relies on w_t being unchanged. Hence,

$\gamma = 0 \rightarrow$ The future young are worse off.

$\gamma = 1 \rightarrow$ The future young are better off.

Problem 2 - recap

For debt-financed reform (small γ):

- **Politics:** The old at t_0 are indifferent but the young support it. Future individuals are hurt by the reform
- **Capital:** Savings increase but capital decreases as it is crowded out by gov. debt.
- **GE effects:** Capital decreases, which hurts future generations.

For tax-financed reform (large γ):

- **Politics:** The old at t_0 are indifferent but the young does not support it. Future individuals benefit from the reform
- **Capital:** Both savings and capital increase.
- **GE effects:** Capital increases, which increase wages and future generations benefit from it.